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Locally Stationary Random Processes

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Abstract

The concept of a locally stationary random process is introduced and the harmonic analyis of such a process is examined. the generalized Wiener-Khintchine relations obeyed by locally stationary random processes are discussed. The dielectric turbulence responsible for beyond-the-horizon scattering of radio waves is cited as an example of a locally stationary random process.

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1. Preliminary considerations

In this note we introduce the concept of a locally stationary random process, which is a natural generalization of the familiar notion of a stationary random process. We shall exhibit a remarkable symmetry between the covariance and the spectral density of a locally stationary random process, which gives insight into the structure of stationary random processes. First of all, we recall some relevant facts about random processes in general, and stationary random processes in particular [1], [2].

Let $x(t)$ be a random process (in general complex) with mean zero. Then by the covariance of $x(t)$ is meant the quantity

$$\Gamma(t,t') = \langle x(t)x^*(t') \rangle , \tag{1}$$

where the angular parentheses denote the ensemble average and the asterisk denotes the complex conjugate. It is apparent from (1) that $\Gamma(t,t') = \Gamma^*(t',t)$. Suppose $x(t)$ is harmonizable, i.e., that $x(t)$ can be written as the Fourier-Stieltjes transform

$$x(t) = \int \exp(i\omega t)\,dg(\omega) \tag{2}$$

of another random process $g(\omega)$, where the integral is to be taken in the mean-square sense. (Here, as below, all integrals are taken from $-\infty$ to ∞.) Then it follows that the covariance of $x(t)$ can be written as

$$\Gamma(t,t') = \int\int \exp\left[i(\omega t - \omega' t')\right] \Psi(\omega,\omega')\,d\omega\,d\omega' , \tag{3}$$

where the two-dimensional spectral density $\Psi(\omega,\omega')$ is related to the increments of the random process $g(\omega)$ by

$$(4) \qquad \Psi(\omega,\omega')d\omega\, d\omega' = \langle dg(\omega) dg^{*}(\omega') \rangle \quad .$$

In writing (3) and (4), we assume the physical case of absolutely continuous spectral densities. Moreover we have

$$(5) \qquad \Psi(\omega,\omega') = \frac{1}{(2\pi)^2} \int\int \exp\left[-i(\omega t - \omega' t')\right] \Gamma(t,t')dt\, dt' \ ,$$

so that $\Gamma(t,t')$ and $\Psi(\omega,\omega')$ are two-dimensional Fourier transform pairs.

The zero-mean random process $x(t)$ is said to be stationary (in the wide sense) if its covariance is of the form

$$\Gamma(t,t') = \langle |x|^2 \rangle \ R(t-t') \ .$$

The quantity $\langle |x|^2 \rangle$ is the average instantaneous power of the process $x(t)$ and is independent of time; $R(t-t')$ is the normalized correlation function of $x(t)$, with $R(0) = 1$. As is well-known, in the absolutely continuous case $R(t-t')$ is the Fourier transform of a non-negative function $\Phi(\omega)$, the power spectral density of $x(t)$, i.e.

$$(6) \qquad R(\tau) = \int \exp(i\omega\tau)\, \Phi(\omega)\, d\omega \ ,$$

$$(7) \qquad \Phi(\omega) = \frac{1}{2\pi} \int \exp(-i\omega\tau)\, R(\tau)\, d\tau \ .$$

Equations (6) and (7), together with the requirements on $R(\tau)$ and $\Phi(\omega)$, namely that $R(\tau)$ be a stationary covariance or correlation function and that $\Phi(\omega)$ be non-negative, are the celebrated Wiener-Khintchine relations. It is to be noted that $\Phi(\omega)$ is normalized, i.e.,

$$\int \Phi(\omega)\, d\omega = 1 .$$

The harmonic decomposition theorem for stationary random processes [1], [2] consists of the statement that the relation between the increments of the random process $g(\omega)$ appearing in (2) and $\Phi(\omega)$ is

$$\langle dg(\omega) dg^{*}(\omega') \rangle = \langle |x|^{2} \rangle \Phi(\omega) \delta(\omega-\omega') d\omega\, d\omega' ,$$

or equivalently

$$\Psi(\omega,\omega') = \Phi(\omega)\, \delta(\omega-\omega') . \tag{8}$$

Equation (8) is sometimes interpreted by saying that in the case of stationary random processes the two-dimensional spectral density $\Psi(\omega,\omega')$ can be localized on the line $\omega = \omega'$ in the (ω,ω')-plane. Because of the properties of the delta function, (8) can just as well be written as

$$\Psi(\omega,\omega') = \Phi\left(\frac{\omega+\omega'}{2}\right) \delta(\omega-\omega') ; \tag{8}$$

the significance of this form will appear below.

2. Locally stationary random processes

Locally stationary random processes are a natural generalization of stationary random processes, and will be defined as follows*. The zero-mean random process $x(t)$ is said to be locally stationary (in the wide sense) if its covariance can be written as

$$\Gamma(t,t') = \left\langle \left| x\left(\frac{t+t'}{2}\right) \right|^2 \right\rangle R(t-t') \ , \tag{10}$$

where as before $R(t-t')$ is a normalized correlation function. The factor $\left\langle \left| x\left(\frac{t+t'}{2}\right) \right|^2 \right\rangle$ represents change of the average instantaneous power with time; we shall assume that

$$\int |x(\tau)|^2 \, d\tau < \infty$$

except for the special case $\langle |x(\tau)^2| \rangle = \text{const.}$, corresponding to stationary processes. The symmetrized variable $\frac{t+t'}{2}$ has been chosen to assure that $\Gamma(t,t') = \Gamma^*(t',t)$ as it must, and because of its suggestive meaning as the average of the points t and t'. Heuristically speaking, equation (10) should be a good approximation if the average instantaneous noise power changes slowly with respect to the characteristic width of the correlation function R. For simplicity, we call the functions appearing in the right side of (10) Γ_1 and Γ_2 respectively, and rewrite (10) as

$$\Gamma(t,t') = \Gamma_1\left(\frac{t+t'}{2}\right) \Gamma_2(t-t') \ . \tag{11}$$

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* In this connection, see the definition of local homogeneity given by A.N. Kolmogoroff [3].

We now come to the main point of this note, which is that if a random process $x(t)$ is locally stationary, so that its covariance $\Gamma(t,t')$ is of the form (11), then the same is true of its two-dimensional spectral density $\Psi(\omega,\omega')$. To see this, we first write (5) in the form

$$\Psi(\omega,\omega') = \frac{1}{(2\pi)^2} \int\int \exp\left[-i\left(\frac{\omega+\omega'}{2}\right)(t-t') - i\left(\frac{t+t'}{2}\right)(\omega-\omega')\right]\Gamma(t,t')dtdt' .$$

Then, introducing the new variables $u = \frac{t+t'}{2}$, $v = t-t'$ and using (11), we find that

$$\Psi(\omega,\omega') = \Psi_1\left(\frac{\omega+\omega'}{2}\right) \Psi_2(\omega-\omega') \tag{12}$$

as asserted, where

$$\Psi_1(w) = \frac{1}{2\pi} \int \exp(-ivw)\, \Gamma_2(v)\, dv \ , \tag{13}$$

$$\Psi_2(z) = \frac{1}{2\pi} \int \exp(-iuz)\, \Gamma_1(u)\, du \ ,$$

and

$$\Gamma_1(u) = \int \exp(iuz)\, \Psi_2(z)\, dz \ , \tag{14}$$

$$\Gamma_2(v) = \int \exp(ivw)\, \Psi_1(w)\, dw \ .$$

Thus the functions Ψ_1 and Γ_2 as well as Ψ_2 and Γ_1 are Fourier transform pairs. It follows that Ψ_1 is non-negative, being the Fourier transform of the correlation function Γ_2, and that Ψ_2 is a correlation function, being the Fourier transform of the non-negative function Γ_1. Note that when Ψ_2 is the delta function, (12) reduces to the spectral density of a stationary random process (see equation (9)).

Equations (11), (12), (13) and (14), together with the stated requirements on the functions Γ_1, Γ_2, Ψ_1, and Ψ_2 constitute a natural generalization of the usual Wiener-Khintchine relations for stationary random processes. Moreover our study of locally stationary random processes has " revealed" the structure of stationary random processes. The delta function in equations (8) and (9) for the two-dimensional spectral density of a stationary random process is the Fourier transform of the constant average power $\langle |x|^2 \rangle$, just as the correlation function of stationary white noise (with constant power spectrum) is the delta function.

3. Conclusions

The concept of a locally stationary random process introduced here may prove of value in technological applications of noise theory, since local stationarity is a much more reasonable assumption to make about the kind of random processes encountered in nature than is stationarity. (However, the usual heuristics of approximating locally stationary random processes in different regions by "pieces" of suitable stationary random processes is undoubtedly sound.) Consider, for example, the problem of the scattering of radio waves by dielectric noise, i.e., by turbulence-induced fluctuations of refractive index. If turbulent mixing operating in gradients of refractive index is accepted as the mechanism responsible for the dielectric noise (as described elsewhere[4],[5],[6]), then it is clear that the power level of the dielectric noise will vary in space and time in proportion to the prevailing local gradients of refractive index. On the other hand, aside from this variable power factor, the local structure of the turbulence can be expected to be the same everywhere, at least if attention is confined to a range of eddy sizes intermediate between the macro-eddy and micro-eddy dimensions. This situation is typical of those to which the notion of locally stationary random processes seems well suited, and indeed it is the physical problem which suggested the notion to the author.

The mathematically inclined reader will find some material relevant to this note in papers by Getoor[7] and Loève[8].

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